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NONLINEAR PROBLEM OF A STEADY-STATE FLOW OF A WEIGHABLE LIQUID BOUNDED BY A FREE SURFACE ABOUT A SYSTEM OF VORTICES

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The problem of a fluid flow with a free surface about a system of two vortices of opposite intensity is considered within the framework of the nonlinear theory. The range of parameters of the problem in which there is no stationary solution is found. Results of the numerical experiment on the effect of vortex intensities and the Froude number on the shape of the free surface and the hydrodynamic reactions of the singularities are given.

The problem of a fluid flow that is bounded by a free surface about a system of hydrodynamic singularities has broad applications. The majority of related studies are devoted to a single vortex in flow (see, e.g., [1] and references therein). The only example of the solution of the problem of a flow about a system of two vortices and sources is given by Val'dman [2]. Great progress in the field of development of the numerical methods of solving nonlinear problems of wave hydrodynamics has increased interest in this problem.

Our goal is to develop a new numerical method of solving nonlinear problems of a free surface-bounded steady-state flow of a weighable liquid about the hydrodynamic singularities. The method is applied to the solution of the flow problem for a system of two vortices of opposite intensity. The effect of the character of wave formation on the wave resistance of the vortices is studied. Great attention is paid to the profiles of the generated waves.

It is noteworthy that the numerical method of solving nonlinear stationary wave problems is a variant of the more general method of integral boundary equations the use of which has recently led to significant achievements in this field of research [3].

1. Let a stationary flow with a free surface L flow about a system of two vortices of opposite intensity. The fluid is assumed to be ideal, incompressible, weighable, and homogeneous. A coordinate system in which the x axis is located along the unperturbed level of the free surface and the y axis passes through the point of position of the first vortex (Fig. 1) is introduced. In this system, the vortex of intensity Γ is located at the point $(0, -h)$, and the vortex of intensity $-\Gamma$ is at the point $(d, -h)$. We introduce the following notation: V_∞ is the fluid velocity on infinity at the left, ρ is the density of the fluid, g is the acceleration of gravity, and $f(x)$ is a function that describes the shape of the free surface L.

We consider the problem in the plane of a complex variable $z = x + iy$. The analytical function $\bar{V}(z)$ that describes the fluid motion should satisfy the conditions of constant pressure and the zero normal velocity component on the free surface:

$$\operatorname{Im}(\bar{V}(z)(1 + if'(x))) = 0, \qquad z = x + if(x), \qquad |x| < +\infty; \tag{1.1}$$

$$f(x) = \frac{1}{2g}(V_\infty^2 - |V(z)|^2), \qquad z = x + if(x), \qquad |x| < +\infty. \tag{1.2}$$

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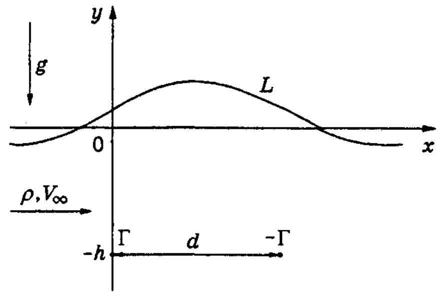

Fig. 1. Flow pattern.

In addition, the conditions of damping of the velocity perturbations and free surface at infinity on the left should be satisfied:

$$\lim_{x\to-\infty} \bar{V}(z) = V_\infty, \qquad \lim_{x\to-\infty} f(x) = 0. \tag{1.3}$$

2. For the functions $\bar{V}(z)$ and $f(x)$, the boundary-value problem (1.1)–(1.3) is nonlinear. The nonlinear nature is due to two factors: 1) the complex velocity $\bar{V}(z)$ enters (1.2) in a nonlinear manner; 2) the shape of the free surface at which the boundary conditions (1.1) and (1.2) are satisfied is unknown. This circumstance creates some difficulties. We reduce the boundary-value problem (1.1)–(1.3) to the solution of a system of integral equations in terms of two real functions of one variable.

We introduce the intensity $\gamma(x)$ of the vortex-layer located on the free surface, assuming that L is a smooth curve, $\gamma(x)$ satisfies the Hölder condition, and $\gamma(-\infty) = 0$. In these assumptions, the complex velocity of the fluid is described by the formula

$$\bar{V}(z) = V_\infty + \frac{\Gamma}{2\pi i}\frac{1}{z-z_1} - \frac{\Gamma}{2\pi i}\frac{1}{z-z_2} + \frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty} \frac{\gamma(\xi)}{z-\zeta(\xi)}\,\frac{1-if'(\xi)}{\sqrt{1+(f'(\xi))^2}}\,d\xi,$$

$$z_1 = -ih, \qquad z_2 = d - ih, \qquad \zeta(\xi) = \xi + if(\xi). \tag{2.1}$$

From the assumption on the damping of the vortex-layer intensity at infinity at the left, it follows that the function $\bar{V}(z)$ constructed according to (2.1) satisfies condition (1.3).

In approaching L from below, the limiting value of the function $\bar{V}(z)$ is determined by the Sokhotskii–Plemelj formula

$$\bar{V}(z) = \bar{V}_0(z) + \frac{1}{2}\gamma(x)\,\frac{1-if'(x)}{\sqrt{1+(f'(x))^2}}, \tag{2.2}$$

where $\bar{V}_0(z)$ is calculated from (2.1) for $z = x + if(x)$; the corresponding improper integral should be understood in the sense of the principal Cauchy value.

With allowance for (2.1) and (2.2), the kinematic and dynamic conditions (1.1) and (1.2) are reduced to

$$\operatorname{Im}(\bar{V}_0(z)(1+if'(x))) = 0, \qquad z = x + if(x); \tag{2.3}$$

$$f(x) = \frac{1}{2g}\Big(V_\infty^2 - V_{0s}^2(z) - \gamma(x)V_{0s}(z) - \frac{1}{4}\gamma^2(x)\Big), \qquad z = x + if(x),$$

$$V_{0s}(z) = \operatorname{Re}\Big(\bar{V}_0(z)\,\frac{1+if'(x)}{\sqrt{1+(f'(x))^2}}\Big), \qquad z = x + i\,f(x). \tag{2.4}$$

Here $V_{0s}(z)$ is the tangent component of the velocity $V_0(z)$.

The hydrodynamic loads R_{xj} and R_{yj} that act on the vortices are determined by the formula

$$R_{yj} + i\,R_{xj} = -\rho_1 \Gamma \Big(V_\infty + \frac{\Gamma}{2\pi i} \frac{1}{z_2 - z_1} + \frac{1}{2\pi i} \int\limits_{-\infty}^{+\infty} \frac{\gamma(\xi)}{z_j - \zeta(\xi)} \frac{1 - i f'(\xi)}{\sqrt{1 + (f'(\xi))^2}}\, d\xi \Big),$$

where $j = 1$ for the vortex of intensity Γ and $j = 2$ for the vortex of intensity $-\Gamma$.

3. To solve the system of nonlinear integral equations (2.3) and (2.4), we use the generalized Newton method [4]:

$$G^{(n)} + G_\gamma^{(n)} \Delta\gamma^{(n)} + G_f^{(n)} \Delta f^{(n)} + G_{f'}^{(n)} \Delta f'^{(n)} = 0; \tag{3.1}$$

$$\Delta f^{(n)} = F^{(n)} + F_\gamma^{(n)} \Delta\gamma^{(n)} + F_f^{(n)} \Delta f^{(n)} + F_{f'}^{(n)} \Delta f'^{(n)}; \tag{3.2}$$

$$\Delta\gamma^{(n)} = \gamma^{(n+1)} - \gamma^{(n)}, \quad \Delta f^{(n)} = f^{(n+1)} - f^{(n)}, \quad \Delta f'^{(n)} = f'^{(n+1)} - f'^{(n)}, \quad n = 0, 1, \ldots;$$

$$\gamma^{(0)}(x) = 0, \qquad f^{(0)}(x) = 0, \qquad f'^{(0)}(x) = 0, \qquad |x| < +\infty. \tag{3.3}$$

Here $G(\gamma, f, f')$ and $F(\gamma, f, f')$ are the left part of (2.3) and the right part of (2.4), respectively, $G^{(n)}$, $G_\gamma^{(n)}$, $G_f^{(n)}$, $G_{f'}^{(n)}$, $F^{(n)}$, $F_\gamma^{(n)}$, $F_f^{(n)}$, and $F_{f'}^{(n)}$ are the integrodifferential operators and their partial derivatives with respect to γ, f, and f' for the functions $\gamma = \gamma^{(n)}$, $f = f^{(n)}$, and $f' = f'^{(n)}$, respectively. The expression (3.3) determines the initial approximation for γ, f, and f'.

We consider the free surface L in the computational domain $[x_a, x_b]$ ($x_a \ll -h$, $x_b \gg h$). It is assumed that $\gamma(x) = f(x) = 0$ for $x \leqslant x_a$, and $f(x)$ and $\gamma(x)$ are assumed to be periodic for $x \geqslant x_b$. Under these assumptions, the expression for the complex velocity (2.1) can be written in the form

$$\bar{V}(z) = V_\infty + \frac{\Gamma}{2\pi i} \frac{1}{z - z_1} - \frac{\Gamma}{2\pi i} \frac{1}{z - z_2} + \frac{1}{2\pi i} \int\limits_{x_a}^{x_b} \frac{\gamma(\xi)}{z - \zeta(\xi)} \frac{1 - i f'(\xi)}{\sqrt{1 + (f'(\xi))^2}}\, d\xi$$

$$+ \frac{1}{2\pi i} \int\limits_{x_b}^{x_b + \lambda} \sum_{k=0}^{+\infty} \frac{\gamma(\xi)}{z - (\xi + k\lambda + i f(\xi))} \frac{1 - i f'(\xi)}{\sqrt{1 + (f'(\xi))^2}}\, d\xi. \tag{3.4}$$

Here the last term is due to the presence of the infinite system of waves of length λ for $x \geqslant x_b$.

We solve the system of linear integral equations (3.1), (3.2) by the method of high-order panels [5]. With this in view, we construct a grid of nodes x_k ($k = 1, \ldots, N+1$), $x_1 = x_a$, and $x_{N+1} = x_b$ in the region $[x_a, x_b]$. We require the fulfillment of the linearized boundary conditions (3.1) and (3.2) at the points of collocation $x_{0k} \in [x_k, x_{k+1}]$ ($k = 1, \ldots, N$). Then, this system is discretized under the assumption that the free surface in the interval $[x_k, x_{k+1}]$ is approximated by a parabola, and the vortex-layer intensity on this interval by a linear function. Here it is assumed that $\gamma(x_1) = f(x_1) = 0$. At the points of collocation, the values of the derivative $f'(x)$ are expressed via the values of $f(x)$ in the adjacent nodes by means of fourth-order numerical differentiation formulas [6]. As a result, we obtain a system of linear algebraic equations relative to $\Delta\gamma^{(n)}(x_k)$ and $\Delta f^{(n)}(x_k)$ ($k = 2, \ldots, N+1$).

Below, we shall describe the algorithm for solving the problem as a whole. The Newton method is iterated until a certain solution is obtained. The initial approximation is determined by (3.3). The last term in (3.4), which is due to the presence of the infinite wave system, is omitted. At each step, a system of $2N$ linear equations is solved by the method of high relaxation. After that, the second iterative process connected with the use of the Newton method as the initial approximation of the already known solution is used. Here the length of the generated waves λ is found from the solution obtained at the previous step of the additional iterative process. This process proceeds until the necessary accuracy is achieved.

4. Using the suggested method, we performed the numerical experiment on the solution of the problem of a steady-state flow of a weighable fluid about a system of two vortices of opposite intensity in the presence of a free surface. The dimensionless parameters of the problem are the Froude number $\mathrm{Fr} = V_\infty/\sqrt{gh}$ and

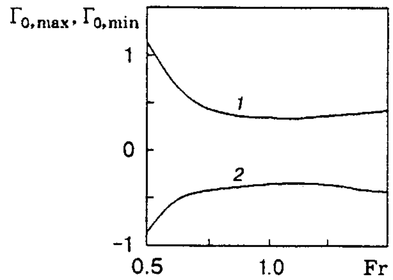

Fig. 2. Limiting values of $\Gamma_{0,\max}$ (1) and $\Gamma_{0,\min}$ (2) at which the solution of the problem is possible versus the Froude number Fr.

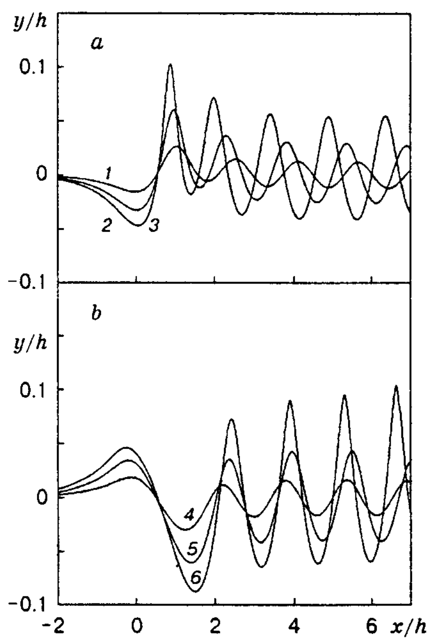

Fig. 3

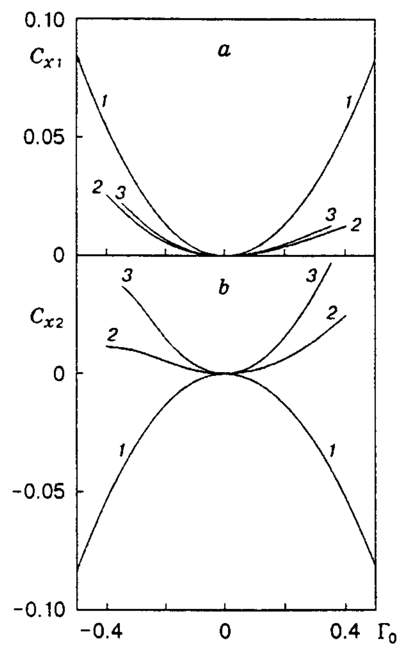

Fig. 4

Fig. 3. Shape of the free surface for Fr = 0.5: curves 1–6 refer to $\Gamma_0 = -0.3$, -0.6, -0.861, 0.4, 0.8, and 1.142, respectively.

Fig. 4. Forces C_{x1} and C_{x2} that act on the vortices versus the intensities Γ_0 of these vortices: curves 1–3 refer to Fr = 0.5, 0.8, and 1, respectively.

TABLE 1

Fr	Γ_0					
	−0.3	−0.2	−0.1	0.1	0.2	0.3
	C_{y1}					
0.5	0.6037	0.4016	0.2004	−0.1996	−0.3985	−0.5968
0.8	0.6141	0.4067	0.2017	−0.1981	−0.3923	−0.5825
1.0	0.6144	0.4068	0.2018	−0.1979	−0.3909	−0.5789
	C_{y2}					
0.5	−0.5971	−0.3987	−0.1997	0.2003	0.4014	0.6033
0.8	−0.5842	−0.3933	−0.1984	0.2015	0.4061	0.6134
1.0	−0.5787	−0.3921	−0.1982	0.2017	0.4065	0.6149

the vortex intensity $\Gamma_0 = \Gamma/(V_\infty h)$. The domain $[-2.5\lambda_0, 4.5\lambda_0]$, in which $\lambda_0 = 2\pi V_\infty^2/g$ is the wavelength obtained according to the linear theory, was chosen as a computational domain. It was assumed in the calculations that $N = 210$ and $d = \lambda_0/2$.

For a fixed value of the Froude number, the level of perturbations of the free surface is determined by the vortex intensities. The height of the free surface cannot be greater than $y_{\rm lim}/h = {\rm Fr}^2/2$. This circumstance imposes restrictions on the value of Γ_0. Increasing or decreasing Γ_0, one can obtain the limiting values of $\Gamma_{0,\max}$ and $\Gamma_{0,\min}$ at which the problem posed can be solved. These values are given in Fig. 2 for various Fr. The maximum rise of the free surface which is calculated for the limiting values of Γ_0 is not smaller than 85–86% of $y_{\rm lim}/h$.

Figure 3 shows results of the numerical experiment obtained with the use of the estimate of the effect of the parameters of the problem on the character of wave formation. In the case where the vortex located upstream has a positive intensity, the maximum value of the free surface is attained at infinity on the right (Fig. 3b). If the vortex of negative intensity is located upstream, the maximum rise of the free surface is observed immediately above the system of vortices (Fig. 3a). Thus, the character of the waves formed in a distant field is completely determined by the intensity sign of the vortex located downstream. At the limiting values of Γ_0 at which the problem can be solved, the distinct nonlinear character of the generated waves is observed.

The results of calculations for the dimensionless coefficients of hydrodynamic loads

$$C_{xj} = \frac{2R_{xj}}{\rho V_\infty^2 h}, \qquad C_{yj} = \frac{2R_{yj}}{\rho V_\infty^2 h} \quad (j = 1, 2)$$

that act on the vortices are given in Fig. 4 and Table 1. For Fr = 0.5, owing to the small distance between the vortices and the small amplitudes of the generated waves, the main contribution to C_{xj} is from the force of interaction between the vortices. This fact explains the difference between these coefficients calculated for different Fr and the presence of the pulling force for the downstream flow for Fr = 0.5. For Fr = 0.8 and 1, the forces of the wave effect, which result in the appearance of the force of resistance for both vortices, are manifested. The values of the coefficient C_{yj} are determined by the Joukowski lift force and depend little on the Froude number. This fact is supported by the values listed in Table 1; for fixed Γ_0 and different Fr, they differ slightly.

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